ALGEBRAIC TOPOLOGY I WS23/24, HOMEWORK SHEET 9

DEADLINE: FRIDAY, DECEMBER 15TH

Problem 1. This problem was incorrect as stated, I apologize for the confusion. Thanks to Phil Pützstück and Oleksandr Kharchenko for making me aware of this.

One can associate to every 1-dimensional real line bundle $p\colon E \to B$ a 2-sheeted covering $Y(E)$ whose fiber over a point b is given by $\pi_0(E_b - \{0\})$. One can also associate to every 2-sheeted covering $q\colon Y \to B$ a 1-dimensional real line bundle $E(Y)$ via the balanced product $\mathbb{R} \times_{\mathbb{Z}/2} Y$, using that $\mathbb{Z}/2$ acts on Y by swapping the two elements in each fibre (the local triviality shows that this a continuous action) and acts on $\mathbb{R}$ via multiplication with ± 1.

Now, $Y(E(Y))$ is again isomorphic to Y by sending a point $y \in Y$ to the equivalence class of $(1, y)$ in the balanced product. But the composite $E(Y(E))$ is in general not isomorphic to E again. For an explicit counterexample, see the post of Tom Goodwillie in

https://mathoverflow.net/questions/106497/non-trivial-vector-bundle-over-non-paracompact-contractible-space

There, a non-trivial line bundle E over the pushout X of

$$\begin{array}{ccc} \mathbb{R}_{>0} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & X \end{array}$$

is constructed by gluing two trivial line bundles $\mathbb{R} \times \mathbb{R}$ along the bundle isomorphism $h\colon \mathbb{R}_{>0} \times \mathbb{R} \xrightarrow{\cong} \mathbb{R}_{>0} \times \mathbb{R}$ sending (x, t) to (x, xt).

However, the isomorphism h is trivial on $\mathbb{R}_{>0} \times \pi_0(\mathbb{R} - \{0\})$, meaning that the associated 2-sheeted covering $Y(E)$ is the trivial one. Hence $E(Y(E))$ is the trivial line bundle, and non-isomorphic to E itself.

In conclusion, the set of isomorphism classes of 2-sheeted coverings is a natural retract of the set of isomorphism classes of 1-dimensional real line bundles. But in general, not all line bundles come from 2-sheeted coverings.

Problem 2. Using a partition of unity, show that any vector bundle over a paracompact base space can be given a Euclidean metric.